3. Real-valued and Vector-valued Functions

In the next few lectures, we will generalize concepts of calculus of real-valued functions of one variable, including limit, continuity, and differentiability, to real-valued and vector-valued functions.

In this lecture, we will discuss

- Real-valued and Vector-valued Functions of Several Variables
  - Definitions
  - Important examples
    - Linear Functions
    - Distance Functions
    - Projection
  - Find the domain and range of a given function
  - Vector fields (are examples of vector-valued functions)
- Graph of a function of Several Variables
  - Definition
  - Level Set
    - Level/Contour curves
    - Level/Contour surfaces

Real-valued and Vector-valued Functions

Recall to describe a function $f : \mathbb{R} \to \mathbb{R}$, we need

- Domain
- Range
- A rule that assigns to each element of the domain a unique element in the range.

Now, let's generalize this to a function $f : \mathbb{R}^m \to \mathbb{R}^n$, where m and n are positive integers.

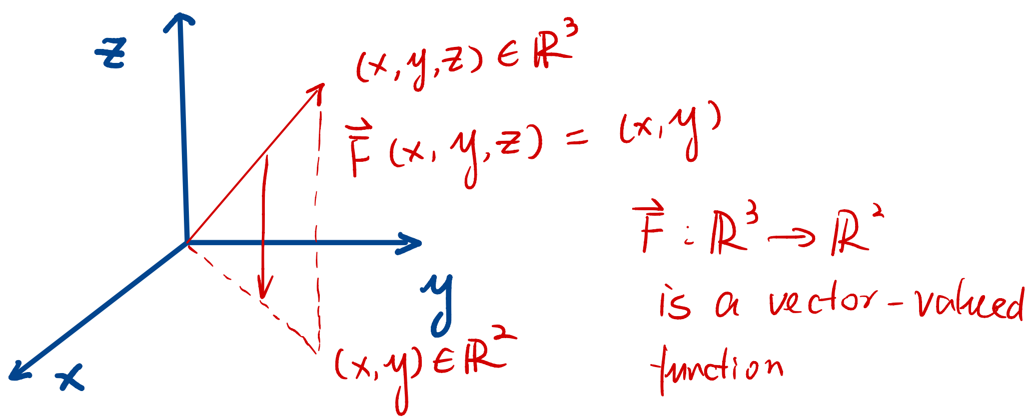
Defintion. Real-Valued and Vector-Valued Functions
A function whose domain is a subset U of $\mathbb{R}^m, m \geq 1$, and whose range is contained in $\mathbb{R}^n$ is called a real-valued function of m variables if $n = 1$, and a vector-valued function of m variables if $n > 1$.

$\vec{F}: \mathbb{R}^n \to \mathbb{R}^m$

Important Examples of Functions

$A_{m\times n}\vec{x} = \vec{b}$ $\qquad$ $\vec{F}(\vec{x}) = A\vec{x} = \vec{b}$

- **Linear Functions**
  - A function of the form $f(x, y) = ax + by + c$, where a, b, and c are constants (i.e., real numbers), is called a *linear function* (of two variables).
  - In general, a *linear function* of n variables is defined on $\mathbb{R}^n$ by the formula $f(x_1, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n + b$, where $a_1, \ldots, a_n$ and b are constants.
- **Distance Functions**
  - The distance function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ measures the distance from the point (x, y, z) to the origin.
  - It is a real-valued function of three variables defined on $U = \mathbb{R}^3$.
- **Projection**
  - A projection $\mathbf{F}(x, y, z) = (x, y)$ is a vector-valued function of three variables that assigns to every vector (x, y, z) in $\mathbb{R}^3$ its projection (x, y) onto the xy-plane.



- **Parametric equations**
  - Recall the *Parametric Equation of a Line* we derived in Lecture 1:

$$\mathbf{l}(t) = \mathbf{a} + t\mathbf{v}, \quad t \in \mathbb{R},$$

Or

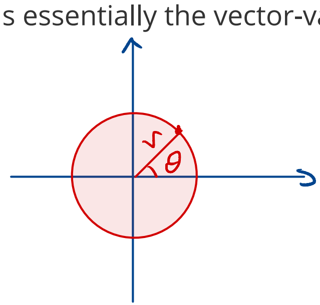
$$\mathbf{l}(t) = (a_1 + tv_1, a_2 + tv_2), \quad t \in \mathbb{R}.$$

Notice $\mathbf{l}$ is a vector-valued function from $\mathbb{R}$ to $\mathbb{R}^2$.

  - *Parametric Equation of the Circle*

$$(x, y) = (r\cos\theta, r\sin\theta)$$

is essentially the vector-valued function $\mathbf{F}(r, \theta) = (r\cos\theta, r\sin\theta)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$.



Example 1

Find the domain and range of the functions given below.

1. $f(x,y,z) = \dfrac{3}{x+y}$

2. $f(x,y,z) = e^{-(x^2+y^2+z^2)}$

ANS: 1 Domain The domain is a subset of $\mathbb{R}^3$, since f is a function of 3 varibles.

Since the denominator $x+y \neq 0$, the domain contains all points in $\mathbb{R}^3$ except points with $y=-x$. i.e.

$$U = \{(x,y,z) \in \mathbb{R}^3 \mid y \neq -x\}.$$

Range : Note $f(x,y,z) = \dfrac{3}{x+y}$.

Let $c \neq 0$ be any nonzero number. then

$$f\left(\tfrac{3}{c}, 0, 0\right) = \frac{3}{\frac{3}{c}+0} = c$$

Thus every nonzero $c \in \mathbb{R}$ is in the range of f.

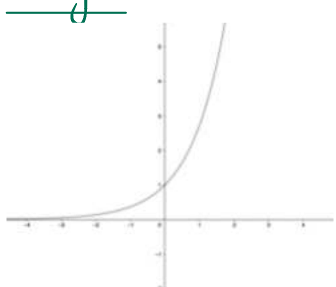
Also, $f(x,y,z) = \frac{3}{x+y} \neq 0$ for any $x,y,z \in \mathbb{R}$.

Thus the range of f is $\mathbb{R}-\{0\}$. (all the real numbers except 0)

2. $f(x,y,z) = e^{-(x^2+y^2+z^2)}$, $f: \mathbb{R}^3 \to \mathbb{R}$

Domain : Since the exponential function e^x is defined for any real number x, the domain is $\mathbb{R}^3$.

Range : Recall the graph of $g(x) = e^x$



Since $-(x^2+y^2+z^2) \leq 0$,

$$0 < e^{-(x^2+y^2+z^2)} \leq 1$$

Thus the range of f is $(0,1]$.

Below, we introduce an important class of vector-valued functions. You might already seen it in your ODE course.

Vector Field

Definition. Vector Field

A *vector field in the plane* is a vector-valued function $\mathbf{F} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, defined on a subset $U \subseteq \mathbb{R}^2$.

A vector field in space is a vector-valued function $\mathbf{F} : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$.

In general, a function $\mathbf{F} : U \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is called a vector field (or, a vector field on U).

Visual representations of vector fields

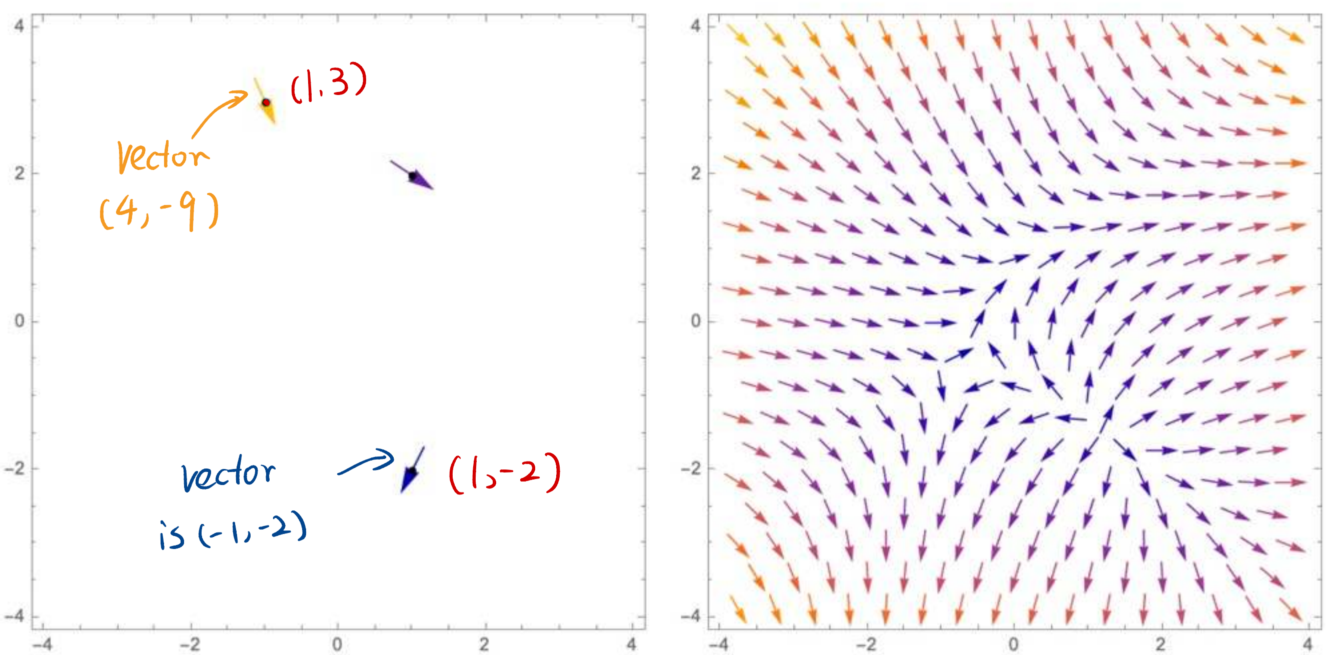
🤔 How to visualize the vector fields?

We often visualize them in the following way:

- Elements of the domain are thought of as points
- The elements of the range are viewed as vectors.

- In $\mathbb{R}^2$,

  Let $f(x, y) = (x^2 + y, 1 + x - y^2)$, then at points $(-1, 3)$, $(1, -2)$ and $(1, 2)$, we draw the corresponding vectors $(4, -9)$, $(-1, -2)$, and $(3, -2)$



consider enough points in $(x,y) \in \mathbb{R}^2$ we have the above vector field.

Digression to related topics in ODE

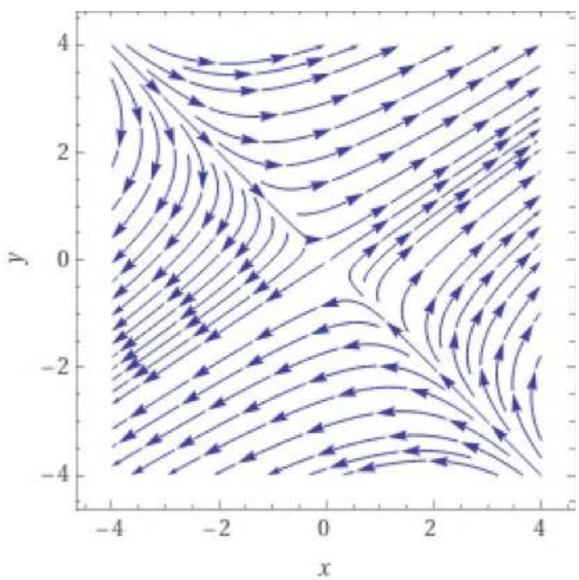
Recall in ODE class, we consider.

$$\begin{cases} x'(t) = 2x + 3y \\ y'(t) = 2x + y \end{cases} \quad \text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

In fact, we have a vector-valued function $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$.

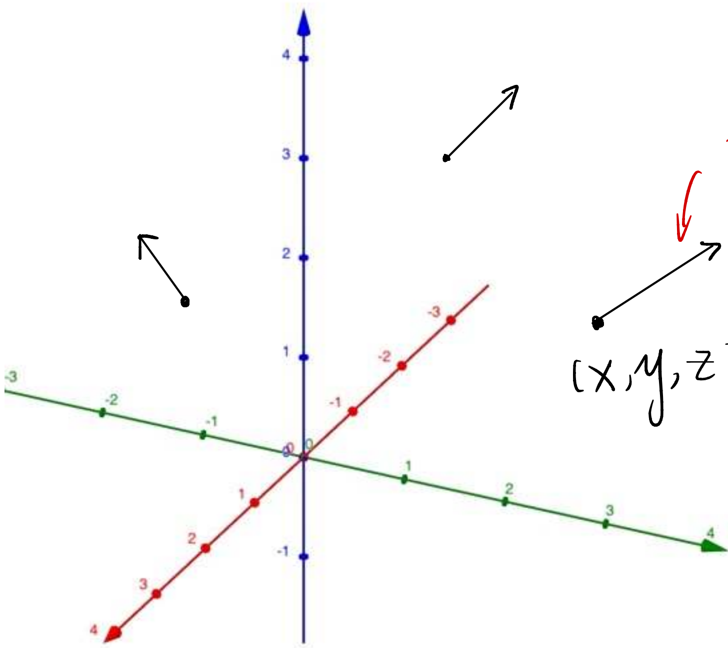
$$\vec{F}(x, y) = (2x + 3y, 2x + y)$$

The vector field can be viewed as



- In $\mathbb{R}^3$,

$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$



vector given by $\vec{F}(x,y,z)$

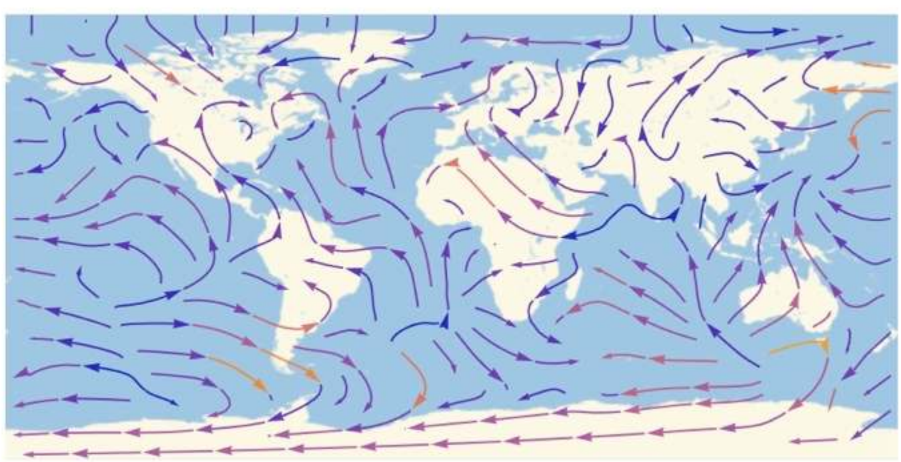
(x,y,z)

- An Example of Application: Wind direction

Plot streamlines for a collection of vectors, and give a geographical range for the domain:

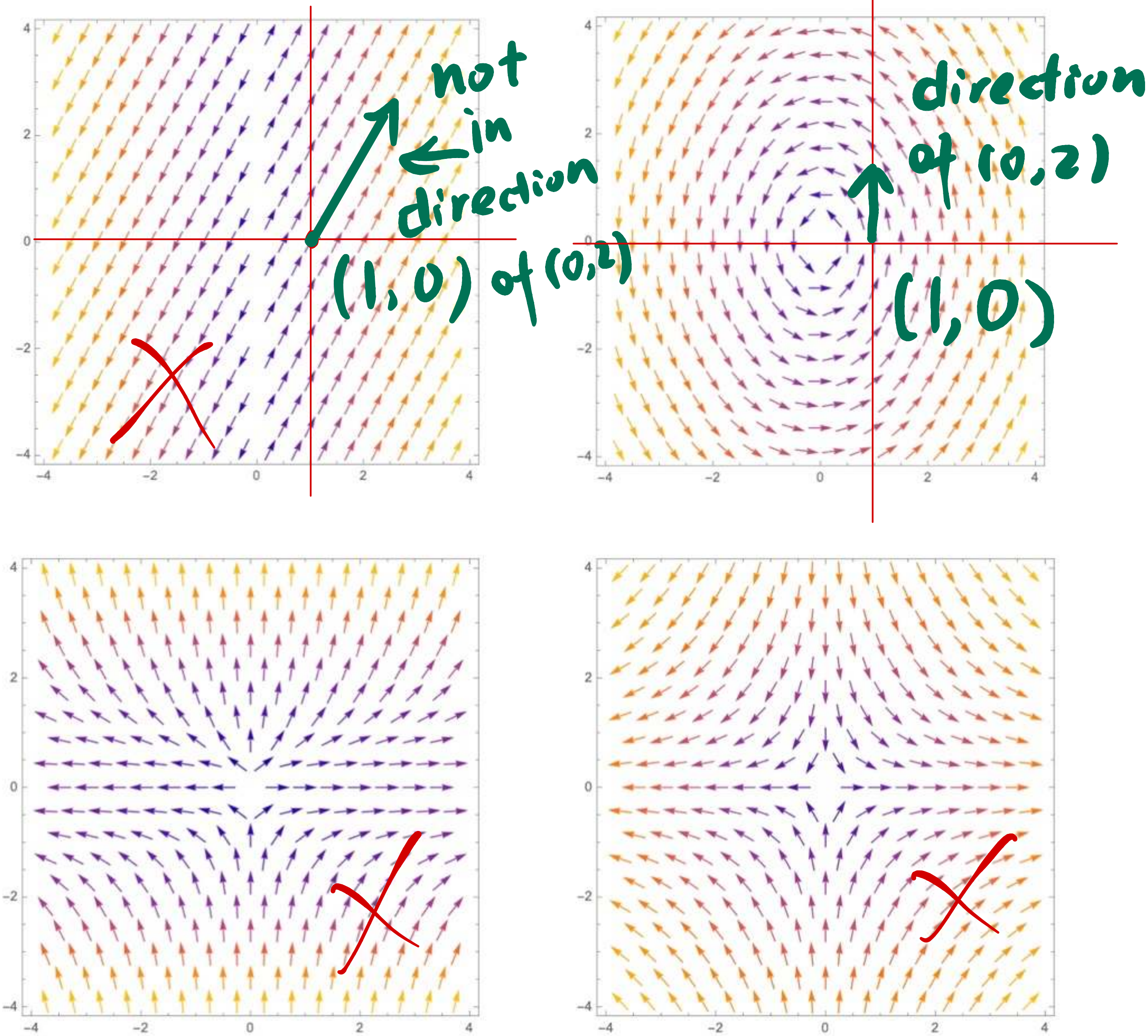
In[1]:= **GeoStreamPlot[** *wind direction* **]**

Out[1]=



At the point $(1,0)$, $\vec{F}(1,0) = (0,2)$

Example 2 Match the planar vector field $\mathbf{F} = \langle -y, 2x \rangle$ with the corresponding plot in the Figures below.

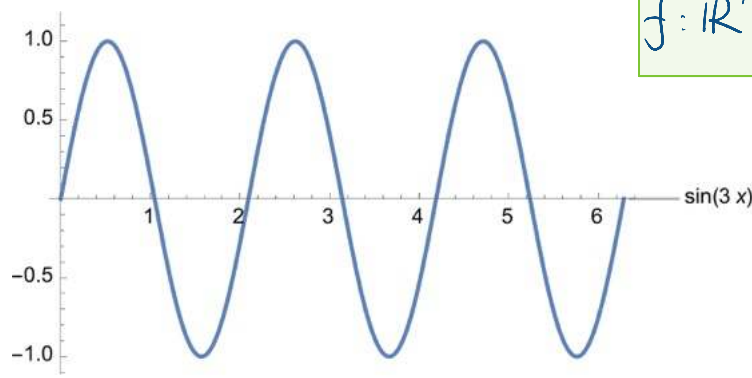


Graph of a function of Several Variables

Recall the graph of a real-valued function $y = f(x)$ of one variable is a curve in the xy-plane.

Each point (x, y) on that curve carries two pieces of data: the value x of the independent variable and the corresponding value $y = f(x)$ of the function.

$f: \mathbb{R}^1 \to \mathbb{R}^1$, $f(x) = \sin 3x$



Alternatively, we can describe the graph of f as the set

$$\boxed{\operatorname{Graph}(f) = \{(x, y) \mid y = f(x) \text{ for some } x \in U\} \subseteq \mathbb{R}^2,}$$

where $U \subseteq \mathbb{R}$ is the domain of f.

We generalize the above description to the following definition.

Definition. Graph of a Real-Valued Function of Two Variables

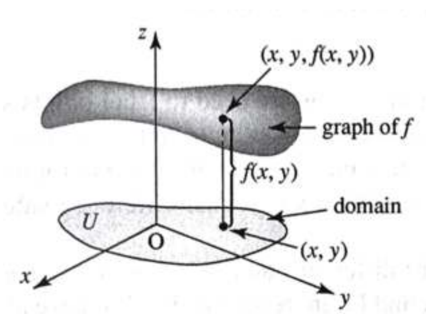
The graph of a real-valued function $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ of two variables is the set

$$\operatorname{Graph}(f) = \{(x, y, z) \mid z = f(x, y) \text{ for some } (x, y) \in U\} \subseteq \mathbb{R}^3,$$

where $U \subseteq \mathbb{R}^2$ denotes the domain of f.

Generally, the graph of a real-valued function $f : U \subseteq \mathbb{R}^m \to \mathbb{R}$ of m variables is the set

$$\operatorname{Graph}(f) = \{(x_1, \ldots, x_m, y) \mid y = f(x_1, \ldots, x_m) \text{ for some } (x_1, \ldots, x_m) \in U\} \subseteq \mathbb{R}^{m+1}.$$



Graph of a function $f: \mathbb{R}^2 \to \mathbb{R}$ ($z = f(x, y)$) is a surface in $\mathbb{R}^3$.

Remark: When $m = 1$ in the above def. we recover the def of Graph (f) for $f: \mathbb{R} \to \mathbb{R}$.

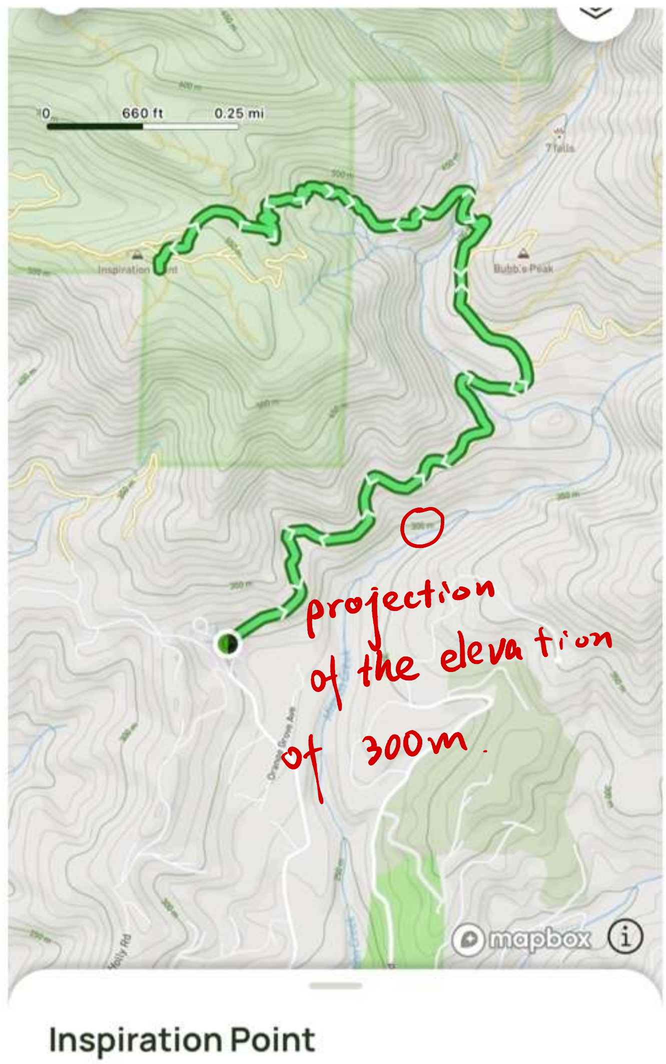
There is another way of visualizing graphs that uses two dimensions to represent the graph of the function $z = f(x, y)$ of two variables.

It consists of drawing level curves (or contour curves), and uses two-dimensional data to obtain three-dimensional information.

For example,

- A contour curve on topographic maps indicates points of the same elevation.
- We find the elevation at various locations
- We can also draw various conclusions: for example, the closer the contour curves are, the steeper the hill; the further apart they become, the smaller the slopes are.

Contour Line Map of the Inspiration Point Trail



Generalizing the above discussion, we have the following:

Definition. Level Set

Let $f : U \subseteq \mathbb{R}^m \to \mathbb{R}$ be a real-valued function of m variables and let $c \in \mathbb{R}$.

The *level set of value c* is the set of all points in the domain U of f on which f has a constant value; that is, Level set of value $c = \{(x_1, \ldots, x_m) \in U \mid f(x_1, \ldots, x_m) = c\}$.

- In particular, for $m = 2$ the level set

$$\{(x, y) \in U \subseteq \mathbb{R}^2 \mid f(x, y) = c\}$$

  is called a *level curve* (of value c) or a *contour curve* (of value c)

- For $m = 3$ the level set

$$\{(x, y, z) \in U \subseteq \mathbb{R}^3 \mid f(x, y, z) = c\}$$

  is called a *level surface* or a *contour surface* (of value c).

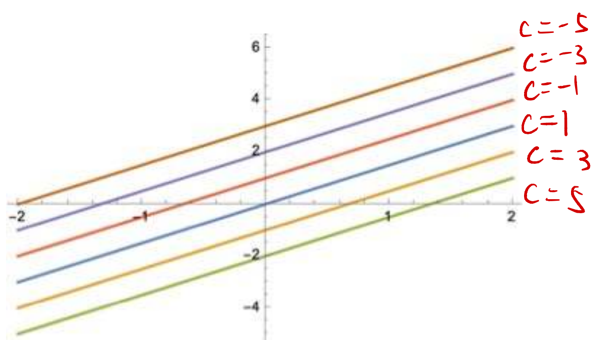
Example 3 Describe the contour diagram of the linear function $f(x, y) = 3x - 2y + 1$.

ANS: Set $f(x,y) = 3x - 2y + 1 = c$, we have

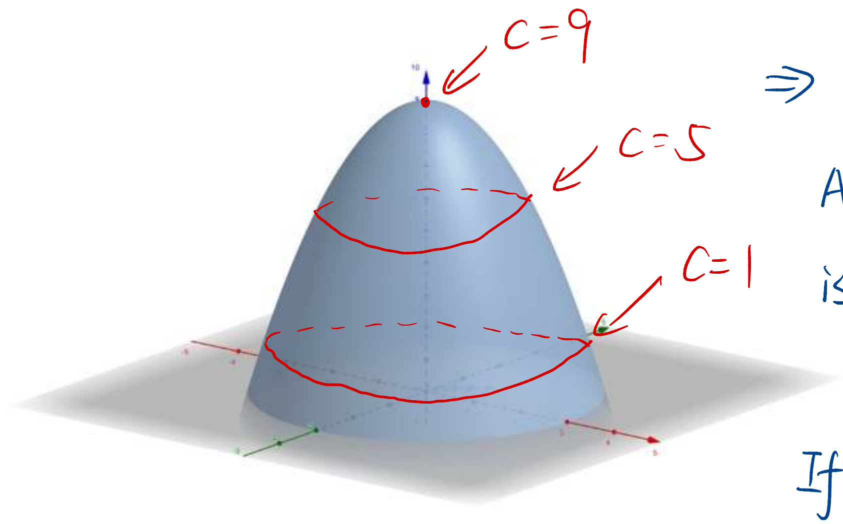
$$3x - 2y = c - 1$$

$$\Rightarrow \quad y = \frac{3}{2}x + \frac{1-c}{2}$$

Thus the level/contour diagrams of f are parallel lines with slope $\frac{3}{2}$.



Example 4 Describe the level curves of $z = 9 - x^2 - y^2$.



ANS: Let $z = f(x,y) = 9 - x^2 - y^2 = c$

$\Rightarrow x^2 + y^2 = 9 - c$

As $9 - c = x^2 + y^2 \geq 0$, there is no curves for

$9 - c < 0$, i.e. $c = 9$

If $c = 9$, we have

$x^2 + y^2 = 9 - c = 0$

$\Rightarrow x = y = 0$

Thus the level curve is a single point if $c = 9$.

If $c < 9$, we have

$x^2 + y^2 = 9 - c > 0$

which is a circle of radis $\sqrt{9-c}$